Tanaka

$F \subset \mathbb{R}^3 \times (0,1) \subset \mathbb{R}^3 \times [0,1]$

surface = link cobordism between $\phi \rightsquigarrow \phi$

$$KJ(F) := |(\Phi_F)_*(1)| \in \mathbb{Z}_{\geq 0}$$

hom. induced on KF homology

If $\chi(F) \neq 0$, $\Rightarrow KJ(F) = 0$

Th 1 [T], [Rasmussen]
$KJ(F) = 2$ for $\forall T^2$-knot F

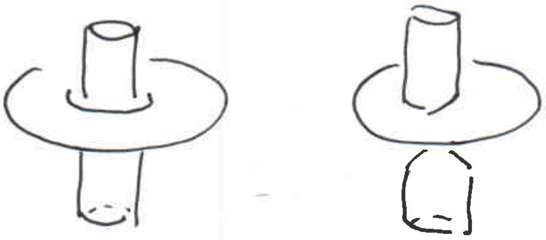
BN(F) generalization of KJ(F)
parameter t $t=0 \to$ original def.

$$BN(F) \in \mathbb{Z}[t] \quad \text{s.t. } BN(F)|_{t=0} = KJ(F)$$

Th 2 [T]
F : surface knot of genus g
(i) g : even $\Rightarrow BN(F) = 0$
(ii) g : odd $\Rightarrow BN(F) = 2^g t^{\frac{g-1}{2}}$

Facts (i) $\forall F$ surface knot of genus g
$\exists h_i^1$ finite number of 1 handles on F
s.t. $F \cup \{h_i^1\}_{i=1}^n$ trivial surface knots
(of genus $g+n$)

(ii) $\forall$ 1 handle is ribbon move equivalent to a trivial 1-handle



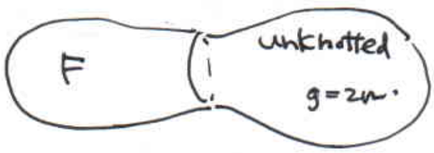
(iii) $F_1 \underset{\text{rib. move}}{\sim} F_2 \Rightarrow BN(F_1) = BN(F_2)$

(proof) $g = 2m+1$ (odd)

① $\exists a \in \mathbb{Z}_{\geq 0}$ s.t. $BN(F) = at^m$

$$1 \longmapsto \pm a t^m$$

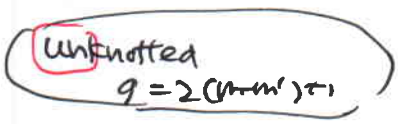
② $m' > 0$



$$1 \longmapsto \longrightarrow \pm a t^m (4t)^{m'}$$

ribbon move equiv.

③ $m' > 0$



$$1 \longmapsto \pm 2 (4t)^{m+m'}$$

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Things to next :

① Compute $KJ(\cdot)$ (or $BN(\cdot)$)

for surface <u>links</u> with $\chi = 0$

② Morrison-Walker's refinement

(removed sign)

$\widetilde{KJ}(\ ) \in \mathbb{Z}$ s.t. $|\widetilde{KJ}| = KJ$

③ Study invariants derived from

the functoriality of sl_n-link homology